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MONOCYCLICITY AND ACYCLICITY OF SECOND ORDER DYNAMIC SYSTEMS

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The problem of monocyclicity and acyclicity of second order dynamic systems is considered. Results of investigations [1, 2] are extended to a wider class of specific dynamic systems.

A similar problem of monocyclicity, and the roughness of periodic solutions of system of differential equations were investigated in [3, 4].

1. We consider in the plane (x, y) the system

$$dx / dt = X(x, y), \quad dy / dt = Y(x, y) \tag{1.1}$$

and introduce the curvilinear system of coordinates (c, φ) by formulas

$$\begin{gathered} y = af(x), \quad [f(x)]^2 + y^2 = c^2 \\ a = \operatorname{tg} \varphi, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad 0 < c < +\infty \end{gathered} \tag{1.2}$$

Let us assume that function f is defined, univalued, and has a continuous first derivative over the set of real numbers $f(0) = 0$ and $f'(x) \neq 0$ for $(x, af(x)) \in \{(x, af(x)): f^2(x) + y^2 > c_0^2\}$, where c_0 is some fixed value of c.

The second and third assumptions imply that for the indicated x function f is strictly monotonic.

We define the inverse transformation of (1.2) by formulas $\varphi = \operatorname{arctg}(y / f(x))$ and $c = [f^2(x) + y^2]^{1/2}$. The Jacobian of transformation (1.2)

$$D(c, \varphi) / D(x, y) = -f'(x) / c \quad (x \neq 0)$$

is of the same sign in each quadrant of the plane (x, y). Hence the inverse transformation ensures one-to-one mapping of the plane (x, y) (without point $(0, 0)$) onto the plane (c, φ) (without point $(0, \varphi)$).

Theorem 1. Let system (1.1) satisfy on the set $D = \{(x, y): f^2(x) + y^2 \geqslant c_0^2\}$ the following conditions:

a) X and Y are continuous and ensure the uniqueness of solution of the Cauchy problem;

b) All points of set D are nonsingular;

c) If $k > 1$, then

$$\begin{gathered} Y(q)X(q_k) \geqslant Y(q_k)X(q) \quad (x \neq 0) \\ Y(0, y)X(0, ky) \geqslant Y(0, ky)X(0, y) \\ f'(kx)Y(q)X(q_k) \geqslant f'(x)Y(q_k)X(q) \quad (x \neq 0) \\ q = (x, af(x)), \quad q_k = (kx, af(kx)) \end{gathered}$$

d) system (1.1) has a limit cycle L_1 on which exists a point (x', y') such that for $k > 1$ the inequalities

$$f'(kx')Y(Q)X(Q_k) > f'(x')Y(Q_k)X(Q)$$
$$f'(x'/k)Y(Q)X(Q_{1/k}) < f'(x')Y(Q_{1/k})X(Q)$$
$$Q = (x', af(x')), \quad Q_k = (kx', af(kx'))$$

are satisfied.

The limit cycle L_1 of system (1.1) is then unique in region D.

P r o o f. Note that the first and third of conditions c) imply tnat $f'(x) > 0$ for $(x, af(x)) \in D$.

Let the equations

$$x_1 = x_1(t), \quad y_1 = y_1(t), \quad t_0 \leqslant t \leqslant T \tag{1.3}$$

define a closed integral curve L_1 of system (1.1). Owing to its closeness it intersects every half-curve of set (1.2) (a branch of curve of set (1.2) issuing from point (0, 0)). Curve L_1 is not tangent to any half-curve of set (1.2). Indeed, if we assume that it is tangent to the half-curve

$$y = f(x) \operatorname{tg} \varphi_0 \tag{1.4}$$

at some point $(x_1(t_1), y_1(t_1))$, then, owing to the continuity of functions (1.3) and the first two of conditions c) follows the existence of neighborhood W of point t_1 such that the section Ω of curve L_1 that correspond to all $t \in W$ coincides with some part of the half-curve (1.4) containing point $(x_1(t_1), y_1(t_1))$. If however, the strict inequality is satisfied for all $t \in W$ under the specified conditions, then owing to the continuous dependence of solution on initial conditions, section Ω, except point $(x_1(t_1), y_1(t_1))$, lies entirely in the part of neighborhood of half-curve (1.4) that correspond to φ which satisfies the inequality $\varphi < \varphi_0$. Hence curve L_1 cannot intersect half-curve (1.4) at the tangency point $(x_1(t_1), y_1(t_1))$. In the presence of the tangency point $(x_1(t_1), y_1(t_1))$ the curve cannot intersect the half-curve (1.4) at any other of its points. Otherwise there would exist a half-curve $y = f(x) \operatorname{tg} \varphi_1$, $0 \leqslant \varphi_1 < \varphi_0 \leqslant 2\pi$ such that some section Ω_1 of curve L_1 is tangent to the latter half-curve at some point $(x_1(t_2), y_1(t_2))$, while being at the same time in a part of the neighborhood of the half-curve that corresponds to a φ such that $\varphi \geqslant \varphi_1$, which is impossible by virtue of the first two conditions of c). Hence the relation

$$f(x_1)Y(x_1, y_1) - y_1 X(x_1, y_1) f'(x_1) \neq 0 \tag{1.5}$$

is satisfied at any point (x_1, y_1) of the closed integral curve L_1.

Let us assume that system (1.1) has in addition to curve L_1, another closed integral curve L_2 defined by the equations $x_2 = x_2(t)$, $y_2 = y_2(t)$, $t_0 \leqslant t \leqslant T$. All points of curve L_2 also satisfy formula (1.5) in which x_2 and y_2 have been substituted for x_1 and y_1. We write down system (1.1) in curvilinear coordinates (c, φ) in the form

$$c^{-1} dc / d\varphi = P(c, \varphi) \tag{1.6}$$
$$P(c, \varphi) = \frac{f(x) f'(x) X(x, y) + yY(x, y)}{f(x) Y(x, y) - yf'(x) X(x, y)}$$

where x and y are functions of c and φ defined by the relations $f(x) = c\cos\varphi$, $y = c\sin\varphi$. Let L_1 and L_2 be defined in coordinates (c, φ) by the equations $c_1 = c_1(\varphi)$, $c_2 = c_2(\varphi)$, $c_1, c_2 > 0$, respectively. By condition a) of the theorem $c_1(\varphi) \neq c_2(\varphi)$, $0 \leqslant \varphi \leqslant 2\pi$. We assume for definiteness that $c_1 > c_2$. From Eq. (1.6) we have

$$J = \int_0^{2\pi} \omega(\varphi)\, d\varphi = 0 \tag{1.7}$$

$$\omega(\varphi) = P(c_1(\varphi), \varphi) - P(c_2(\varphi), \varphi)$$

We represent the integrand in (1.7) in the form

$$\omega(\varphi) = c_1 c_2 E \,/\, (F_1 F_2)$$
$$E = f'(x_1) Y(r_2) X(r_1) - f'(x_2) Y(r_1) X(r_2)$$
$$F_i = f(x_i) Y(r_i) - y_i X(r_i) f'(x_i)$$

where $r_1(\varphi) = (x_1, y_1)$, $r_2(\varphi) = (x_2, y_2)$ are points on the limit curves.

In conformity with condition (1.5) the denominator of the obtained fraction does not vanish. Let us show that the numerator of that fraction is nonnegative. Indeed, it is possible to indicate for every point $r_1 = (x_1, af(x_1)) \in L_1$ $(x_1 \neq 0)$, $(0, y_1) \in L_1$ such two numbers $k > 1$ and $n > 1$ that point

$$r_2 = (x_1 / k,\ af(x_1 / k)) \in L_2, \quad (0, y_1 / n) \in L_2$$

corresponds one-to-one to that point. These points satisfy the second of inequalities in d) and the third of conditions c) whose sign changes to the opposite when $1/k$ is substituted for k. This means that the numerator is nonnegative and, consequently, retains its sign for $0 \leqslant \varphi \leqslant 2\pi$, $\omega(\varphi)$, which contradicts condition (1.7). If follows from this that system (1.1) has no limit cycles imbedded in the limit cycle L_1. Using (1.5), the third of conditions in c) and d), we similarly prove the absence of limit cycles surrounding L_1.

Remarks. 1°. If in Theorem 1 we assume, instead of conditions d), the existence in region D of an open unbounded curve Γ with points as close as desired to the curve $f^2(x) + y^2 = c_0^2$ at all of whose points the first of conditions in d) is satisfied, then system (1.1) has not more than one limit cycle in D, if all remaining conditions of Theorem 1 are satisfied. Indeed, let that condition be satisfied. If a limit cycle of system (1.1) exists, it necessarily must intersect curve Γ at a point where the conditions d) of Theorem 1 are satisfied and, consequently, that limit cycle is unique.

2°. The Theorem 1 in [4] is the corollary of the present theorem, if in the latter we set $f(x) = x$ and take into account Remark 1°.

3°. If in Theorem 1 we reject condition d), a ring region surrounding the boundary of region D at whose every point (x, y), the relationship

$$Y(q) X(q_k) = Y(q_k) X(q)$$
$$f'(kx) Y(q) X(q_k) = f'(x) Y(q_k) X(q)$$

is satisfied is possible. In other words that region may be completely filled with closed integral curves of system (1.1).

4°. If system (1.1) has a limit cycle L_1, then it is orbitally stable. The orbital instability of the periodic solution is established by reversing the inequality signs in c) and d).

T h e r o e m 2. Let us assume that

e) functions $X(x, y) = y$ and $Y(x, y) = -g(x) + y\Phi(x, y)$ satisfy condition in a) and b) of Theorem 1;

f) if $k > 1$

$$\Phi(q) \geqslant \Phi(q_k) \quad (x \neq 0), \quad \Phi(0, y) \geqslant \Phi(0, ky)$$
$$g'(kx)\, Y(q)\, ag(kx) \geqslant g'(x)\, Y(q_k)\, ag(x)$$

g) on the limit cycle L_1 of system (1.1) there is a point (x', y') such that

$$g'(kx')\, Y(Q)\, ag(kx') > g'(x')\, Y(Q_k)\, ag(x')$$
$$g'(x'/k)\, Y(Q)\, ag(x'/k) < g'(x')\, Y(Q_{1/k})\, ag(x')$$

h) the properties of function g are identical to those of function f, then system (1.1) has the single limit cycle L_1 in D.

P r o o f. Let us show that functions X and Y satisfy the remaining conditions in c) and d) of Theorems 1. The fulfilment of the third of conditions in c) and of condition in d) of the theorem is evident. We shall prove the validity of the first and second of condition in c) by contradiction. Let us assume that

$$Y(q)\, X(q_k) - Y(q_k)\, X(q) < 0$$

From the conditions of the theorem we have

$$[-g(x) + ag(x)\quad \Phi(q)]\, ag(kx) - [-g(kx) + ag(kx)\quad \Phi(g_k)]\, ag(x) < 0$$

Taking into account that $a = y / g(x)$ we obtain the inequality

$$y^2 g(kx)\, [\Phi(q) - \Phi(q_k)] / g(x) < 0 \quad (y \neq 0)$$

But $g(x)\, g(kx) > 0$, hence $\Phi(q) < \Phi(q_k)$ which contradicts the first two of conditions f) of Theorem 2. The theorem is proved.

Note that the theorem is valid for any doubly-connected region D^* which represents the plane (x, y) from which the singly-connected region that contains point $(0, 0)$ and is bounded by a closed curve, when all conditions of Theorem 1 are satisfied in D^*.

In Theorem 2, as in Theorem 1, the stable orbital periodic solution is considered when it exists. By reversing the signs of inequalities in f) and g) we establish instability of the periodic solution.

2. Let us establish the conditions of difference between monocyclic and acyclic systems of the second order. We assume that

i) $X = X_1 + X_2, \quad Y = Y_1 + Y_2$

j) the auxilliary system

$$dx / dt = X_1(x, y), \quad dy / dt = Y_1(x, y)$$

satisfies the conditions of uniqueness of solution of the Cauchy problem, and has, as its first inegral $F(x, y) = C$, where $C \geqslant 0$, $F(0, 0) = 0$, and F is a single-valued function with $F(q) < F(q_k)$ in the plane (x, y);

k) in region $D^* = \{(x, y): c_0^2 \leqslant F(x, y) \leqslant c_1^2\}$ (where $c_1 = +\infty$) is admissible) X and Y satisfy all conditions of Theorem 1 with allowance for Remark 1°.

Theorem 3. Let in D^* exist a single closed curve L^* surrounding curve

$$F(x, y) = c_0^2 \tag{2.1}$$

Curve L^* is defined by the equation

$$X_1(x, y)\, Y_2(x, y) - X_2(x, y)\, Y_1(x, y) = 0 \tag{2.2}$$

The expression $X_1Y_2 - X_2Y_1$ changes its sign when passing across curve (2.2). In that case system (1.1) has a single steady limit cycle in D^*.

Proof. The equation $F(x, y) = C$ defines a topographical system of closed curves surrounding the singular point $(0, 0)$, imbedded in each other, and filling the whole area. The derivative F_t' by virtue of system (1.1) is defined by

$$F_t' = F_x'X_2 + F_y'Y_2$$

Then (2.2) is the contact curve of the auxilliary and the (1.1) systems. From j) and k), and conditions of this theorem follows the existence of a single stable limit cycle. The theorem is proved.

If the signs of inequalities in c) and d) of Theorem 1 are reversed with $k > 1$ and all conditions of Theorem 3 satisfied, system (1.1) has a single unstable limit cycle in D^*.

If curve (2.2) intersects curve (2.1), or lies entirely in region D^* without encircling curve (2.1), or degenerates into a point, or is imaginary, there exists an open integral curve passing through the whole region D^*, and containing points as close to the boundary (2.1) as desired. By condition a) of Theorem 1 system (1.1) has no limit cycle in D^*.

Theorem 4. Let us make the following assumptions.

l) in the region

$$D_1 = \left\{(x, y): y^2 + 2\int_0^x g(x)\, dx \geqslant c_0^2\right\}$$

system (1.1) satisfies all conditions of Theorem 2 with allowance for Remark 1°.

m) the equation $\Phi(x, y) = 0$ determines the closed curve which envelops the boundary (2.1) of region D_1. Then system (1.1) has a single stable limit cycle in region D_1.

n) if the closed curve $\Phi(x, y) = 0$ intersects the boundary of D_1, it is either entirely located in D_1, without enveloping that boundary, or degenerates into a point, or is imaginary, then system (1.1) is acyclic in D_1.

Proof. As the topographic system we shall consider the integral curves $F(x, y^2) = C$ of the auxilliary system

$$dx / dt = y, \quad dy / dt = -g(x)$$

By virtue of system (1.1) the derivative F_t' is of the form $F_t' = y^2\Phi(x, y)$. Hence Theorem 4 is a corollary of Theorem 3.

3. A suitable selection of the expansion of $Y(x, y) = -g(x) + y\Phi(x, y)$ makes it possible to determine the criteria of the single limit cycle existence. We apply the results of investigations in Sects. 1 and 2 to the system

$$dx / dt = y, \quad dy / dt = Y(x, y) \tag{3.1}$$

$$Y(x, y) = \sum_{i+k=1}^{3} \alpha_{ik} x^i y^k, \quad \alpha_{ik} = \text{const}$$

Theorem 5. System (3.1) has a single stable limit cycle if the coefficients α_{ik} satisfy the conditions

$$\alpha_{10} < 0, \quad \alpha_{30} < 0, \quad \alpha_{20} = 0 \tag{3.2}$$

$$\delta = \alpha_{21}\alpha_{03} - {}^1/_4\alpha_{12}^2 > 0 \tag{3.3}$$

$$\alpha_{01} > 0, \quad \alpha_{21} + \alpha_{03} < 0 \tag{3.4}$$

$$2\alpha_{10}\lambda^2 + \alpha_{30}\lambda^4 + 2c_0^2 \geqslant 0, \quad \lambda^2 < c_0^2 \tag{3.5}$$

$$\lambda^2 = x_0^2 + y_0^2, \; x_0 = ({}^1/_4\alpha_{12}\alpha_{02} - {}^1/_2\alpha_{03}\alpha_{11}) / \delta$$

$$y_0 = ({}^1/_4\alpha_{12}\alpha_{11} - {}^1/_2\alpha_{21}\alpha_{02}) / \delta$$

Proof. We select function g and Φ of the form

$$g(x) = -\alpha_{10}x - \alpha_{30}x^3$$

$$\Phi(x, y) = \alpha_{01} + \alpha_{11}x + \alpha_{02}y + \alpha_{12}xy + \alpha_{21}x^2 + \alpha_{03}y^2$$

It follows from (3.3) and (3.4) that the equation $\Phi(x, y) = 0$ determines an ellipse surrounding the singular point $(0, 0)$ of system (3.1), hence there exists a c_0 such that the ellipse encircles the curve

$$y^2 + 2\int_0^x g(x)\,dx = c_0^2 \tag{3.6}$$

$$|y| \leqslant c_0, \quad |x| \leqslant \{\alpha_{30}^{-1}[-\alpha_{10} - (\alpha_{10}^2 - 2\alpha_{30}c_0^2)^{1/2}]\}^{1/2}$$

The region D_1 in which condition m) of Theorem 4 is satisfied, has been thus constructed. All conditions of Theorem 2 are also satisfied in D_1. The conditions of Theorem 5 imply that the equation $z = \Phi(x, y)$ defines a paraboloid whose vertex is projected to point (x_0, y_0) of the plane (x, y). Let us show that point (x_0, y_0) lies inside region B bounded by the closed curve (3.6). Let us consider the circle $x^2 + y^2 = \lambda^2$ on which lies point (x_0, y_0). The distance $d = (x^2 + y^2)^{1/2}$ from point (x, y) to the coordinate origin determined by (3.6) is

$$d^2 = \alpha_{30}x^4 / 2 + (\alpha_{10} + 1)x^2 + c_0^2 \geqslant \lambda^2 \tag{3.7}$$

The upper bound of solution of this inequality for $|x|$ exceeds the upper bounds of solution of inequalities (3.6) with conditions (3.5) satisfied. It follows from this that condition (3.5) implies the fulfilment of condition (3.7) at point B and that point $(x_0, y_0) \in B$. Hence function $\Phi(x, y)$ is strictly decreasing when passing in region

D_1 from point q to point q_k. The first of conditions f) of Theorem 2 is then satisfied in that region, from which follows the strict inequality in the first of conditions c) of Theorem 1. To prove this it is sufficient to show that $x \neq 0$ when $k > 1$

$$0 < g'(x) < g'(kx) \tag{3.8}$$

Actually, since $g''(x) = -6\alpha_{30}x$, hence $g''(x) > 0$ when $x > 0$, and the inequality (3.8) is satisfied. Let $x < 0$, then on the axis $y = 0$ point kx lies to the left of point x. Moreover, $g''(x) < 0$, hence condition (3.8) is satisfied for $x < 0$. The last condition of Theorem 2 for function $g(x)$ is verified in the same manner. Thus, region D_1 is monocyclic.

In the first of inequalities in condition (3.4) is reversed, with all remaining conditions of this theorem retained, system (3.1) becomes acyclic, since for $\alpha_{01} < 0$ curve $\Phi(x, y) = 0$ does not encircle point $(0, 0)$. The latter shows that c_0 is such that the curve of contacts lies entirely in region D_1 without encircling its boundary. From this follows the validity of statement n) of Theorem 4. The theorem is proved.

If conditions (3.2), (3.3), and one of conditions

$$\Delta = 4\alpha_{21}\alpha_{03}\alpha_{01} + \alpha_{11}\alpha_{12}\alpha_{02} - \alpha_{11}^2\alpha_{03} - \alpha_{12}^2\alpha_{01} - \alpha_{02}^2\alpha_{21} = 0$$
$$1/4\Delta(\alpha_{21} + \alpha_{03}) > 0$$

which implies that the curve $\Phi(x, y) = 0$ is, respectively, imaginary or degenerates into a point, then system (3.1) is acyclic.

We assume that in the last theorem $\alpha_{20} \neq 0$, and instead of (3.2) consider the conditions $\alpha_{10} < 0$, and $\alpha_{20}^2 - 3\alpha_{10}\alpha_{30} < 0$. We furthermore stipulate that point $(-\alpha_{20}/(3\alpha_{30}), y)$ must not belong to region $\{(x, y)\colon g^2(x) + y^2 \geqslant c_0^2\}$. Then, altering suitably conditions (3.5) and retaining the remaining conditions of Theorem 5, we obtain the confirmation of the existence of a single stable limit cycle in system (3.1).

We note in conclusion that Theorem 1 remains valid also, when in the last of conditions c) and in conditions d) of that theorem the signs are reversed, since such reversal does not alter the sign of the numerator of fraction $\omega(\varphi)$.

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